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A 2-factor with Short Cycles Passing Through Specified Independent Vertices in Graph

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Abstract. For a graph G, we define $\sigma_2(G) := min\{d(u)+d(v)|u, v \notin E(G), u \neq v\}$. Let $k \geq 1$ be an integer and G be a graph of order $n \geq 3k$. We prove if $\sigma_2(G) \geq n + k - 1$, then for any set of k independent vertices $v_1, \ldots, v_k$, G has k vertex-disjoint cycles $C_1, \ldots, C_k$ of length at most four such that $v_i \in V(C_i)$ for all $1 \leq i \leq k$. And show if $\sigma_2(G) \geq n+k-1$, then for any set of k independent vertices $v_1, \ldots, v_k$, G has k vertex-disjoint cycles $C_1, \ldots, C_k$ such that $v_i \in V(C_i)$ for all $1 \leq i \leq k$, $V(C_1) \cup \cdots \cup V(C_k) = V(G)$, and $|C_i| \leq 4$ for all $1 \leq i \leq k - 1$.

The condition of degree sum $\sigma_2(G) \geq n + k - 1$ is sharp.

Key words. Short cycle, degree sum condition, 2-factor, specified vertices.

1. Introduction

In this paper, we only consider finite undirected graphs without loops or multiple edges. We will follow standard terminology and notation from [1] except as indicated. Let $G = (V(G), E(G))$ be a graph, the minimum degree of G will be denoted by $\delta(G)$ and $\sigma_2(G) := min\{d(u) + d(v)|u, v \in V(G), uv \notin E(G), u \neq v\}$ is the minimum degree sum of nonadjacent vertices.(when G is a complete graph, we define $\sigma_2(G) = \infty$.) For $v \in V(G)$ and $U, W \subset V(G)$. We let $N_G(v, U)$ (or simply $N(v, U)$) denote the neighborhood of v in U, i.e., $N(v, U) := \{u \in U|uv \in E(G)\}$. Define $d_G(v, U)$ (or simply $d(v, U)$) denote the degree of v, thus $d(v, U) = |N(v, U)|$. When $U = V(G)$, we simply write $N(v) = N(v, V(G))$ and $d(v) = d(v, V(G))$.

Short cycles is a cycle of length at most four.

Wang [11] considered the degree sum condition, and proved the following:

Theorem 1. *Let G be a graph of order at least $3k$ where k is a positive integer, suppose that $\sigma_2(G) \geq 4k - 1$. Then G contains k vertex-disjoint cycles.*

Egawa et al. [4, 5] considered k vertex-disjoint cycles covering vertices of G. They proved the following results respectively:

Theorem 2. *Let k, d and n be three integers with $k \geq 3$, $d \geq 4k - 1$, $n \geq 3k$. If G is a graph of order n satisfying the condition that $\sigma_2(G) \geq d$. Then G has k vertex-disjoint cycles covering at least $min\{d, n\}$ vertices of G.*

Theorem 3. *Let n, h be integers with $n \geq 6$ and $h \geq 7$. Let G be a graph of order n, and suppose that $\sigma_2(G) \geq h$, then G contains two disjoint cycles C_1 and C_2 such that $|V(C_1)| + |V(C_2)| \geq min\{h, n\}$.*

The following was conjectured in [12] and proved for $k = 2$ in [12] and for all $k \geq 2$ in [6].

Theorem 4. *Let k be an integer with $k \geq 2$, if G is a graph of order $n \geq 4k - 1$ satisfying the condition that $\sigma_2(G) \geq n + 2k - 2$, then for any k independent edges $e_1, \ldots, e_k$ of G, G has k vertex-disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$ for each $i \in \{1, \ldots, k\}$ and $V(C_1) \cup \cdots \cup V(C_k) = V(G)$.*

The degree condition is sharp.

For any graph G, F is a 2-factor of G if and only if F is a union of vertex disjoint cycles that span $V(G)$. The following result by Ore [10] is classic theorem about hamiltonian graphs.

Theorem 5. *Let G be a graph of order $n \geq 3$. If $\sigma_2(G) \geq n$, then G is hamiltonian.*

Ore's theorem implies that G has a 2-factor consisting of exactly one cycle. Brandt et al. [2] and Ralph J. Faudree et al. [7] considered the 2-factor of a graph. They proved the following theorems respectively:

Theorem 6. *Let k be a positive integer and let G be a graph of order $n \geq 4k$. If $\sigma_2(G) \geq n$, then G has a 2-factor with exactly k vertex-disjoint cycles.*

Theorem 7. *Let G be a hamiltonian graph of order $n \geq 6$ and minimum degree at least $5n/12 + 2$. Then G has a 2-factor with two components.*

For a bipartite graph G with partite sets V_1 and V_2, we define $\sigma_{1,1}(G) = min\{d_G(x) + d_G(y) | x \in V_1, y \in V_2, xy \notin E(G)\}$. (When G is a complete bipartite graph, we define $\sigma_{1,1}(G) = \infty$).

Matsumura [9] proved the maximum number of 4-cycle passing through given edges in a graph:

Theorem 8. *Suppose $k \geq 1$, $1 \leq s \leq k$, $n \geq 2k$, and $\sigma_{1,1}(G) \geq max\{\lceil \frac{4n+2s-1}{3} \rceil, \lceil \frac{2n-1}{3} \rceil + 2k\}$. Then for any k independent edges $e_1, \ldots, e_k$ of G, G contains k vertex-disjoint cycles $C_1, \ldots, C_k$ such that $e_i \in E(C_i)$, $|C_i| \leq 6$, and there are at least s 4-cycle in $\{C_1, \cdots, C_k\}$.*

For the problem of cycle passing through specified vertices, Egawa et al. [3] considered k cycles passing through k distinct vertices. They proved the following:

Theorem 9. *Let G be a graph of order n with minimum degree $\delta(G)$. If for a positive integer k,*

a) $n = 3k, \delta(G) \geq 7k - 2/3$ *or*
b) $3k + 1 \leq n \leq 4k, \delta(G) \geq 2n + k - 3/3$ *or*
c) $4k \leq n \leq 6k - 3, \delta(G) \geq 3k - 1$ *or*
d) $n \geq 6k - 3, \delta(G) \geq n/2,$

then for any set of k specified vertices $\{v_1, v_2, \cdots v_k\}$ *there is a 2-factor of G with k cycles* C_i *such that* $v_i \in V(C_i)$ *for* $1 \leq i \leq k$. *The assumption on the minimum degree is sharp in all cases.*

Ishigami [8] discussed the minimum degree condition of G containing k vertex-disjoint cycles of length at most four each of which contains one of the k prescribed vertices, and proved the following Theorem:

Theorem 10. *Let* $k \geq 1$ *be an integer and G a graph of order* $n \geq 3k$ *with* $\delta(G) \geq \lfloor\sqrt{n + k^2 - 3k + 1}\rfloor + 2k - 1$. *Then for any k distinct vertices* $\{x_1, x_2, \cdots x_k\}$, *there exists k vertex-disjoint cycles* $C_1, \ldots, C_k$ *of order at most four with* $x_i \in V(C_i)$ *for* $i \in \{1, \ldots, k\}$.

We consider the minimum degree sum of nonadjacent vertices, and obtain the following:

Theorem 11. *Let* $k \geq 1$ *be an integer and G be a graph of order* $n \geq 3k$ *satisfying the condition that* $\sigma_2(G) \geq n+k-1$. *Then for any set of k independent vertices* $v_1, \ldots, v_k$, *G has k vertex-disjoint cycles* $C_1, \ldots, C_k$ *of length at most four such that* $v_i \in V(C_i)$ *for all* $1 \leq i \leq k$.

The condition $\sigma_2(G) \geq n + k - 1$ is sharp.

Theorem 12. *Let* $k \geq 1$ *be an integer and G be a graph of order* $n \geq 3k$ *satisfying the condition that* $\sigma_2(G) \geq n + k - 1$. *Then for any set of k independent vertices* $v_1, \ldots, v_k$, *G has k vertex-disjoin cycles* $C_1, \ldots, C_k$ *such that* $v_i \in V(C_i)$ *for all* $1 \leq i \leq k$, $V(C_1) \cup \cdots \cup V(C_k) = V(G)$, *and* $|C_i| \leq 4$ *for all* $1 \leq i \leq k - 1$.

The condition $\sigma_2(G) \geq n + k - 1$ is sharp.

2. Examples

The degree condition of Theorem 11 and Theorem 12 are sharp in the following sense.

Example 1. Suppose $n = 3k$. Consider three vertex disjoint graphs G_1, G_2 and G_3. Let G_1 be independent vertex set of order k, G_2 be a complete graph of order $2k - 1$, $G_3 = \{w\}$. Join G_1 completely to G_2, and join G_2 completely to G_3. Thus we get graph G. Then $min\{d(x) + d(y)|xy \notin E(G), x \in V(G_1), y \in v(G_1)\} = n + k - 2$. Clearly, every cycle passing through some vertex of G_1 must contain at least two vertices of G_2. So for the k independent vertices of G_1, G has no k cycles satisfy the property of Theorem 11.

Example 2. Suppose $n \geq 3k$. Consider three vertex disjoint graphs G_1, G_2 and G_3. Let $G_1 = \{x\}$ be a vertex, G_2 be independent vertex set of order k , G_3 be a complete graph of order $n - k - 1$. Join x completely to G_2, and join G_2 completely to G_3. Thus we get graph G. Then $min\{d_G(x) + d_G(y)|xy \notin E(G), x \in V(G_1), y \in V(G_3)\} = k + k + n - k - 2 = n + k - 2$. Clearly, every cycle passing through x must contain at least two vertices in G_2. Therefore for k independent vertices in G_2, G has no k cycles satisfy the property of Theorem 12.

3. Lemmas

Lemma 1. *Let $P = u_1u_2 \cdots u_s$ be a path in G, $u \in V(G) - V(P)$, when $uu_1 \notin E(G)$, if $d(u_s, P) + d(u, P) \geq s$, then G has a path P' with vertex set $V(P') = V(P) \cup \{u\}$ whose end vertices are u and u_1. when $uu_1 \in E(G)$, if $d(u_s, P) + d(u, P) \geq s + 1$, then G has a path P' with vertex set $V(P') = V(P) \cup \{u\}$ whose end vertices are u and u_1.*

Proof. When $uu_1 \notin E(G)$, let $I = \{u_{i-1}|uu_i \in E(G), 1 < i \leq s\}$. $N(u_s, P) \subseteq V(P - u_s)$, $I \subseteq V(P - u_s)$. This implies that $|I \cap N(u_s, P)| \geq |I| + |N(u_s, P)| - |I \cup N(u_s, P)| \geq d(u_s, P) + d(u, P) - (s - 1) \geq s - s + 1 = 1$. It follows that there exists u_{i-1} in $I \cap N(u_s, P)$. Then $P' = uu_iu_{i+1} \cdots u_su_{i-1} \cdots u_1$is the desired path. When $uu_1 \in E(G)$, the result is obvious. □

Lemma 2. *Let $P = u_1u_2 \cdots u_s$ be a path with $s \geq 3$ in G. If $d(u_s, P) + d(u_1, P) \geq s$, then G has a cycle C with $V(C) = V(P)$.*

Proof. Clearly, we may assume that $u_1u_s \notin E(G)$. Let $I = \{u_{i-1}|u_1u_i \in E(G), 1 \leq i < s\}$. Then $I \subseteq V(P) - \{u_s\}$, $N(u_s, P) \subseteq V(P) - \{u_s\}$. This implies that $|I \cap N(u_s, P)| \geq |I| + |N(u_s, P)| - |I \cup N(u_s, P)| \geq d(u_1, P) + d(u_s, P) - (s - 1) \geq s - s + 1 = 1$. It follows that there exists u_{i-1} in $I \cap N(u_s, P)$. Then $P' = u_1u_iu_{i+1} \cdots u_su_{i-1} \cdots u_1$is the desired cycle. □

4. Proof of Theorem 11

We choose G to be a maximal counterexample, that is, if x and y are nonadjacent vertices in G, then $G + xy$ contains k vertex disjoint cycles $C_1, \ldots, C_k$ of length at most four in G such that $v_i \in V(C_i)$ for all $1 \leq i \leq k$. We may assume that $xy \in E(C_k)$. Then $C_1, \ldots, C_{k-1}$ are vertex disjoint cycles of length at most four in G such that $v_i \in V(C_i)$ for all $1 \leq i \leq k - 1$, $v_k \notin \bigcup_{i=1}^{k-1} V(C_i)$, and $\sum_{i=1}^{k-1} |V(C_i)| \leq n - 3$. Among all possible choices of a set of $k - 1$ vertex disjoint cycles of length at most four in G satisfying $v_i \in V(C_i)$ for all $1 \leq i \leq k - 1$, $v_k \notin \bigcup_{i=1}^{k-1} V(C_i)$, $\sum_{i=1}^{k-1} |V(C_i)| \leq n - 3$. Select one collection such that

$$\sum_{i=1}^{k-1} |V(C_i)| \text{is minimum.} \tag{1}$$

Subject to (1), we may further choose $C_1, C_2, \ldots, C_{k-1}$such that

$$\sum_{i=1}^{k-1} d(v_k, C_i) \text{is as small as possible.} \tag{2}$$

Let $L = G[\bigcup_{i=1}^{k-1} V(C_i)]$, $H = G - L$.

We also assume that in this selection any permutation of the vertices $\{v_1, v_2 \cdots, v_k\}$ can be used.

We claim $xv_k \in E(G)$ for all $x \in V(H)$.

Suppose $xv_k \notin E(G)$. Then $d(v_k, H) + d(x, H) \leq |V(H)| - 1$. For otherwise there is a cycle of length four containing v_k in H. Thus $d(v_k, L) + d(x, L) \geq n + k - 1 - (|V(H)| - 1) = |L| + k = \sum_{i=1}^{k-1}(|C_i| + 1) + 1$. This implies there is C_i in L such that $d(v_k, C_i) + d(x, C_i) \geq |C_i| + 2$. If $|C_i| = 3$, let $C_i = x_1x_2v_ix_1$, then $d(x, C_i) = 3$, $d(v_k, C_i) = 2$ ($v_kv_i \notin E(G)$). Thus there is a cycle $C_i' = xx_2v_ix$ and $d(v_k, C_i') = 1$. Which contradicts (2). So $|C_i| = 4$. By (1) and $v_kv_i \notin E(G)$, it is easy to check that $d(v_k, C_i) \leq 2$, thus $d(x, C_i) \geq 4$. We may get a smaller cycle than C_i, a contradiction to (1). Therefore $xv_k \in E(G)$ for all $x \in V(H)$. As claimed.

The proof of the Theorem 11 is divided into three cases:

Case 1. $|H| = 3$. Let $V(H) = \{w_1, w_2, v_k\}$.

As $w_1w_2 \notin E(G)$, then $d(w_1, L) + d(w_2, L) \geq n + k - 1 - 2 = |L| + k = \sum_{i=1}^{k-1}(|C_i| + 1) + 1$. This implies that there exists C_i in L, say $C_i = C_1$, such that $d(w_1, C_1) + d(w_2, C_1) \geq |C_1| + 2$.

If $|C_1| = 4$, say $C_1 = v_1x_1x_2x_3v_1$, then by (1) $N(w_1, C_1) = N(w_2, C_1) = \{x_1, x_2, x_3\}$. Moreover we again by (1) get $v_kx_1 \notin E(G)$, $v_kx_2 \notin E(G)$, $v_kx_3 \notin E(G)$. Let $L_1 = L - C_1$. Then $d(v_1, L_1) + d(v_k, L_1) \geq n + k - 1 - 2 - 2 = |L_1| + k + 2 = \sum_{i=2}^{k-1}(|C_i| + 1) + 4$. This implies that there exists C_i in L_1, say $C_i = C_2$, such that $d(v_1, C_2) + d(v_k, C_2) \geq |C_2| + 2$. By (1) $|C_2| \neq 4$, we get $|C_2| = 3$. And as $d(v_k, C_2) \leq 2$, $d(v_1, C_2) \leq 2$, that is $d(v_1, C_2) + d(v_k, C_2) \leq 4$, a contradiction to $d(v_1, C_2) + d(v_k, C_2) \geq 5$.

Hence $|C_1| = 3$. $d(w_1, C_1) + d(w_2, C_1) \geq |C_1| + 2 = 5$, say $C_1 = v_1x_1x_2v_1$. We may assume without loss of generality that $d(w_1, C_1) = 3$, $d(w_2, C_1) \geq 2$ and $w_2x_2 \in E(G)$, if $v_kx_2 \in E(G)$, then there exist cycle $v_kx_2w_2v_k$ and cycle $v_1x_1w_1v_1$. So $v_kx_2 \notin E(G)$. By $d(w_2, C_1) \geq 2$, if $w_2x_1 \in E(G)$, then $v_kx_1 \notin E(G)$. Similarly, if $w_2v_1 \in E(G)$, then $v_kx_1 \notin E(G)$. So $d(v_k, C_1) = 0$. Let $L_1 = L - C_1$. Then $d(v_1, L_1) + d(v_k, L_1) \geq n + k - 1 - 6 = |L_1| + k - 1 = \sum_{i=2}^{k-1}(|C_i| + 1) + 1$. This implies there exists C_i in L_1, say $i = 2$, such that $d(v_1, C_2) + d(v_k, C_2) \geq |C_2| + 2$. By (1) $|C_2| \neq 4$, $|C_2| = 3$. As $d(v_k, C_2) \leq 2$, $d(v_1, C_2) \leq 2$, that is $d(v_1, C_2) + d(v_k, C_2) \leq 4$, a contradiction to $d(v_1, C_2) + d(v_k, C_2) \geq 5$.

Case 2. $|H| = 4$. Let $V(H) = \{w_1, w_2, w_3, v_k\}$.

As $d(w_1, L) + d(w_2, L) \geq n + k - 1 - 2 = n - 4 + k + 1 = |L| + k + 1 = \sum_{i=1}^{k-1}(|C_i| + 1) + 2$. This means that there exists C_i in L, say $i = 1$, such that $d(w_1, C_1) +$

$d(w_2, C_1) \geq |C_1| + 2$. If $|C_1| = 4$, say $C_1 = v_1x_1x_2x_3v_1$. By (1) $N(w_1, C_1) = N(w_2, C_1) = \{x_1, x_2, x_3\}$. Hence $v_kx_1 \notin E(G)$, $v_kx_2 \notin E(G)$, $v_kx_3 \notin E(G)$. Let $L_1 = L - C_1$. Then $d(v_1, L_1) + d(v_k, L_1) \geq n + k - 1 - 3 - 3 = |L_1| + k + 1 = \sum_{i=2}^{k-1}(|C_i| + 1) + 3$. This implies that there exists C_i in L_1, say $i = 2$, such that $d(v_1, C_2) + d(v_k, C_2) \geq |C_2| + 2$. By (1) $|C_2| \neq 4$, $|C_2| = 3$. As $d(v_k, C_2) \leq 2$, $d(v_1, C_2) \leq 2$, that is $d(v_1, C_2) + d(v_k, C_2) \leq 4$ which is a contradiction.

Hence $|C_1| = 3$. Say $C_1 = x_1x_2v_1x_1$, $L_1 = L - C_1$. As $d(w_1, C_1) + d(w_2, C_1) \geq 5$. We may assume $d(w_1, C_1) = 3$. If $w_2x_1 \in E(G)$, then $w_3x_1 \notin E(G)$, $v_kx_1 \notin E(G)$. If $w_2x_2 \in E(G)$, $w_2x_1 \in E(G)$, then $w_3x_1 \notin E(G)$, $w_3x_2 \notin E(G)$, $v_kx_1 \notin E(G)$, $v_kx_2 \notin E(G)$. If $w_3x_1 \in E(G)$, then $v_kx_1 \notin E(G)$. So $d(w_2, C_1) + d(w_3, C_1) + d(v_k, C_1) \leq 5$, $d(w_2, L_1) + d(w_3, L_1) + d(v_1, L_1) + d(v_k, L_1) \geq 2(n + k - 1) - 15 = 2|L_1| + 2(k - 2) + 1 = 2\sum_{i=2}^{k-1}(|C_i| + 1) + 1$. This implies that there exists C_i in L_1, say $i = 2$, such that $d(w_2, C_2) + d(w_3, C_2) + d(v_1, C_2) + d(v_k, C_2) \geq 2(|C_2| + 1) + 1 = 2|C_2| + 3$.

If $|C_2| = 4$, then by (1), $d(v_1, C_2) \leq 2$, $d(v_k, C_2) \leq 2$, $d(w_2, C_2) \leq 3$, $d(w_3, C_2) \leq 3$, that is $d(w_2, C_2) + d(w_3, C_2) + d(v_k, C_2) + d(v_1, C_2) \leq 10$, a contradiction.

Hence $|C_2| = 3$. Say $C_2 = v_2y_1y_2v_2$. That is $d(w_2, C_2) + d(w_3, C_2) + d(v_1, C_2) + d(v_k, C_2) \geq 9$. On the other hand $d(v_1, C_2) \leq 2$, $d(v_k, C_2) \leq 2$. So $d(w_2, C_2) + d(w_3, C_2) \geq 5$. We may assume without loss of generality $d(w_3, C_2) = 3$. If $d(w_2, C_2) = 3$, then $d(v_1, C_2) + d(v_k, C_2) \geq 3$. If $v_ky_1 \in E(G)$, then we get two cycles $v_ky_1w_3v_k$, $v_2y_2w_2v_2$. So $v_ky_1 \notin E(G)$. If $v_ky_2 \in E(G)$, then we get two cycles $v_ky_2w_3v_k$, $v_2y_1w_2v_2$. So $v_ky_2 \notin E(G)$. So $d(v_k, C_2) = 0$. Thus we obtain $d(v_1, C_2) + d(v_k, C_2) \leq 2$ a contradiction. Hence $d(w_2, C_2) \leq 2$. It is not difficult to see that $d(w_2, C_2) = 2$, $d(v_1, C_2) = 2$, $d(v_k, C_2) = 2$. If $w_2y_1 \in E(G)$, then we get two cycles $v_ky_1w_2v_k$, $v_2y_2w_3v_2$. If $w_2y_2 \in E(G)$, then we get two cycles $v_ky_2w_2v_k$, $v_2y_1w_3v_2$. So $w_2y_1 \notin E(G)$, $w_2y_2 \notin E(G)$, $d(w_2, C_2) \leq 1$, a contradiction to $d(w_2, C_2) = 2$.

Case 3. $|H| \geq 5$.

Let w_1, w_2, w_3 and $w_4 \in V(H)$. Then $d(w_1, L) + d(w_2, L) + d(w_3, L) + d(w_4, L) \geq 2n + 2k - 2 - 4 = 2n + 2k - 6 \geq 2(|L| + 5) + 2k - 6 = 2(\sum_{i=1}^{k-1}(|C_i| + 1)) + 6$. This implies there is C_i in L such that $d(w_1, C_i) + d(w_2, C_i) + d(w_3, C_i) + d(w_4, C_i) \geq 2(|C_i| + 1) + 1 = 2|C_i| + 3$.

If $|C_i| = 3$. Then $d(w_1, C_i) + d(w_2, C_i) + d(w_3, C_i) + d(w_4, C_i) \geq 9$. Say $C_i = x_1x_2v_ix_1$. We may assume without loss of generality $d(w_1, C_i) = 3$, $d(w_2, C_i) \geq 2$ and $d(w_3, C_i) \geq 1$. If $d(w_2, C_i) = 3$, then $d(w_3, C_i) \leq 1$. Since suppose $w_3x_2 \in E(G)$, then there are two cycles $v_kw_3x_2w_2v_k$, $w_1v_ix_1w_1$. So $w_3x_2 \notin E(G)$. Suppose $w_3x_1 \in E(G)$, then there are two cycles $v_kw_3x_1w_2v_k$, $w_1v_ix_2w_1$. So $w_3x_1 \notin E(G)$. Similarly $d(w_4, C_i) \leq 1$. That is $d(w_1, C_i) + d(w_2, C_i) + d(w_3, C_i) + d(w_4, C_i) \leq 8$, a contradiction. Hence $d(w_2, C_i) = 2$, i.e., $d(w_3, C_i) + d(w_4, C_i) \geq 4$. If $N(w_2, C_i) = \{x_1, x_2\}$, then $w_3x_1 \notin E(G)$, $w_3x_2 \notin E(G)$. Similarly, $w_4x_1 \notin E(G)$, $w_4x_2 \notin E(G)$. That is $d(w_3, C_i) + d(w_4, C_i) \leq 2$, a contradiction. Therefore we may assume $N(w_2, C_i) = \{x_2, v_i\}$, then $w_3x_2 \notin E(G)$, $w_4x_2 \notin E(G)$. Furthermore, if $w_3x_1 \in E(G)$, then $w_4x_1 \notin E(G)$. Since there are two cycles $v_kw_3x_1w_4v_k$, $w_1v_ix_2w_1$. That is $d(w_3, C_i) + d(w_4, C_i) \leq 3$, a contradiction.

Therefore $|C_i| = 4$. Then $d(w_1, C_i) + d(w_2, C_i) + d(w_3, C_i) + d(w_4, C_i) \geq 11$. Let $C_i = x_1x_2x_3v_ix_1$. By (1), we may assume $d(w_1, C_i) = 3$, $d(w_2, C_i) = 3$,

$d(w_3, C_i) = 3$, that is $N(w_1, C_i) = N(w_2, C_i) = N(w_3, C_i)$. We obtain two cycles $v_k w_2 x_2 w_3 v_k$, $w_1 x_1 v_i x_3 w_1$. This completes the proof of Theorem 11.

5. Proof of Theorem 12

We assume that G does not have k cycles satisfying the property of Theorem 12. By Theorem 11, we can choose vertex disjoint cycles $C_1, \ldots, C_k$ such that

(i) $v_i \in V(C_i)$ for all $1 \leq i \leq k$.
(ii) $|C_i| = 3$, $i \in I_m$ for some $I_m \subset \{1, \ldots, k\}$,$|I_m| = m$ and that m is maximal, that is to say, for any $j \in \{1, \ldots, k\} - I_m$, $|C_j| \neq 3$.
(iii) The length of a longest path in $G - T$ is maximal.

Where $T := G[\bigcup_{i=1}^{k} V(C_i)]$. Let $P = u_1 \cdots u_s$ be a longest path in $G - T$, $t = n - |T|$.

Claim 1. $t = s$.

Suppose $t > s$. For any $u \in V(G) - T - V(P)$. It is obvious $uu_1 \notin E(G)$, $uu_s \notin E(G)$. And by Lemma 1, If $d(u, P) + d(u_1, P) \geq s$, then there is a path P' with vertex set $V(P) \cup \{u\}$, which contradicts to (iii). Hence $d(u, P) + d(u_1, P) \leq s - 1$, $d(u, P) + d(u_s, P) \leq s - 1$. Thus $2d(u, T) + d(u_1, T) + d(u_s, T) \geq 2n + 2k - 2 - 2(s - 1) - 2(t - s - 1) = 2k + 2|T| + 2 = 2\sum_{i=1}^{k}(|C_i| + 1) + 2$. Therefore there is some C_i, say $i = 1$, satisfying $2d(u, C_1) + d(u_1, C_1) + d(u_s, C_1) \geq 2(|C_1| + 1) + 1 = 2|C_1| + 3$.

If $|C_1| = 4$, say $C_1 = v_1 w_1 w_2 w_3 v_1$. By the maximality of m, $d(w, C_1) \leq 3$, $w \in \{u_1, u_s, u\}$. So $2d(u, C_1) \geq 5$, i.e., $d(u, C_1) = 3$. As $d(u_1, C_1) + d(u_s, C_1) \leq 6$, $2d(u, C_1) \geq 5$, i.e., $d(u, C_1) = 3$. $d(u_1, C_1) + d(u_s, C_1) \geq 5$. By the symmetry of u_1 and u_s, we may assume $d(u_1, C_1) = 3$, then $d(u_s, C_1) \geq 2$. If $u_s v_1 \in E(G)$, then $u_s w_1 \notin E(G)$, $u_s w_3 \notin E(G)$, $u_s w_2 \in E(G)$. Then we replace C_1 and P by $C_1' = u_1 w_1 v_1 w_3 u_1$ and $P' = u_2 \cdots u_s w_2 u$, $|P'| = |P| + 1$, a contradiction to (iii). So $u_s v_1 \notin E(G)$. If $u_s w_1 \in E(G)$, $u_s w_3 \in E(G)$. Then we replace C_1 and P by $C_1' = u_s w_1 v_1 w_3 u_s$ and $P' = u_{s-1} \cdots u_1 w_2 u$, $|P'| = |P| + 1$, a contradiction to (iii). So we may assume $u_s w_1 \in E(G)$, $u_s w_2 \in E(G)$. Then we replace C_1 and P by $C_1' = u_1 w_1 v_1 w_3 u_1$ and $P' = u_2 \cdots u_s w_2 u$, $|P'| = |P| + 1$, a contradiction to (iii).

So $|C_1| = 3$. Say $C_1 = v_1 w_1 w_2 v_1$. $2d(u, C_1) + d(u_1, C_1) + d(u_s, C_1) \geq 9$. By $d(u_1, C_1) + d(u_s, C_1) \leq 6$, $2d(u, C_1) \geq 3$, i.e., $d(u, C_1) \geq 2$. We say $d(u, C_1) \neq 3$. If $d(u, C_1) = 3$, then $d(u_1, C_1) + d(u_s, C_1) \geq 3$, we may assume $d(u_1, C_1) \geq 2$, by the symmetry of w_1 and w_2, assume $u_1 w_1 \in E(G)$. Then we replace C_1 and P by $C_1' = u v_1 w_2 u$, $P' = w_1 u_1 \cdots u_s$, $|P'| = |P| + 1$, a contradiction. So $d(u, C_1) = 2$, $d(u_1, C_1) + d(u_s, C_1) \geq 5$, then we may assume $uw_2 \in E(G)$, $d(u_1, C_1) = 3$, $d(u_s, C_1) \geq 2$. If $uv_1 \in E(G)$, then we replace C_1 and P by $C_1' = u v_1 w_2 u$, $P' = w_1 u_1 \cdots u_s$, $|P'| = |P| + 1$, a contradiction. So $uw_1, uw_2 \in E(G)$. As $d(u_s, C_1) \geq 2$, assume $u_s w_2 \in E(G)$, then we replace C_1 and P by $C_1' = u_1 v_1 w_1 u_1$, $P' = u_2 \cdots u_s w_2 u$, $|P'| = |P| + 1$, a contradiction. As claimed.

Claim 2. $G[V(P)]$ is hamiltonian.

Suppose $G[V(P)]$ is not hamiltonian. Then $u_1 u_s \notin E(G)$. By Lemma 2, $d(u_1, P) + d(u_s, P) \leq s - 1$. Thus $d(u_1, T) + d(u_s, T) \geq n + k - 1 - s + 1 = n - s + k =$

$\sum_{i=1}^{k}(|C_i|+1)$. So there exists some C_i, say $i = 1$, satisfying $d(u_1, C_1)+d(u_s, C_1) \geq |C_1| + 1$. If $|C_1| = 3$, then $d(u_1, C_1) + d(u_s, C_1) \geq 4$, implying that there exists a hamilton cycle C_1' of $G[V(C_1) \cup V(P)]$ containing v_1. Thus $C_1', C_2, \ldots, C_k$ are the desired cycles, a contradiction. So $|C_1| = 4$. By the maximality of m, assume $d(u_1, C_1) = 3$, $d(u_s, C_i) \geq 2$. It is easily seen that there is also a hamilton cycle C_1' of $G[V(C_1) \cup V(P)]$ containing v_1, and $C_1', C_2, \ldots, C_k$ are the desired cycles, a contradiction. As claimed.

Therefore we may assume $u_1u_s \in E(G)$.

Case 1. $d(u_i, V(C_j)) \leq 1$ for any $i, j (i \leq s, j \leq k)$.

As $\sigma_2(G) \geq n + k - 1$, G is $(k + 1)$- connected, $G - \{v_1, \ldots, v_k\}$ is connected. Hence there is some i, j satisfying $d(u_i, C_j - \{v_j\}) \geq 1$, say $i = 1, j = 1$.

If $|C_1| = 4$, say $C_1 = w_1w_2w_3v_1w_1$.

If $u_1w_1 \in E(G)$, then $u_sw_2 \notin E(G)$. Or else say $C_1' = u_1w_1v_1w_3w_2u_s \cdots u_1$, thus $C_1', C_2, \ldots, C_k$ are the desired cycles, a contradiction. Similarly, $u_sv_1 \notin E(G)$. If $u_1w_2 \in E(G)$, by the similar argument, we get $u_sw_3 \notin E(G)$, $u_sw_1 \notin E(G)$. So $d(u_s, C_1) = 0$. And by lemma 1, if $d(w_2, P) + d(u_s, P) \geq s + 1$, there exists a hamilton path w_2Ru_1 with vertex set $V(P) \cup \{w_2\}$ connecting two end vertices w_2 and u_1, yielding the cycle $C_1' = u_1w_1v_1w_3w_2Ru_1$. Thus $C_1', C_2, \ldots, C_k$ are the desired cycles, a contradiction. So $d(w_2, P) + d(u_s, P) \leq s$. Then $d(u_s, C_1) + d(w_2, C_1) = d(w_2) + d(u_s) - [d(w_2, P) + d(u_s, P)] - d(w_2, T - C_1) - d(u_s, T - C_1) \geq n + k - 1 - s - \sum_{i=2}^{k}|C_i| - (k - 1) = n - s - \sum_{i=2}^{k}|C_i| = |C_1|$. So $d(u_s, C_1) \geq |C_1| - 2 = 2$, a contradiction to $d(u_s, C_1) = 0$. Hence $u_1w_2 \notin E(G)$. Then by Lemma 1, $d(w_2, P) + d(u_s, P) \leq s - 1$. And $d(u_s, C_1) + d(w_2, C_1) \geq n+k-1-(s-1)-\sum_{i=2}^{k}|C_i|-(k-1) = |C_1|+1$. So $d(u_s, C_1) \geq |C_1|+1-2 = 3$, a contradiction to $d(u_s, C_1) \leq 2$.

So $u_1w_1 \notin E(G)$. By the symmetry of w_1 and w_3, $u_1w_3 \notin E(G)$. Hence $u_1w_2 \in E(G)$. Then $u_sw_3 \notin E(G)$, $u_sw_1 \notin E(G)$. $d(u_s, C_1) \leq 2$. Then by lemma 1, $d(w_3, P)+d(u_s, P) \leq s-1, d(u_s, C_1)+d(w_3, C_1) \geq n+k-1-(s-1)-\sum_{i=2}^{k}|C_i|-(k - 1) = |C_1| + 1$. So $d(u_s, C_1) \geq |C_1| + 1 - 2 = 3$, a contradiction.

Therefore $|C_1| = 3$. Say $C_1 = w_1w_2v_1w_1$. By the symmetry of w_1 and w_2, assume $u_1w_1 \in E(G)$, then $u_sw_2 \notin E(G)$, $u_sv_1 \notin E(G)$, $d(u_s, C_1) \leq 1$. If $u_1w_2 \in E(G)$, then $u_sw_1 \notin E(G)$, $d(u_s, C_1) = 0$. By Lemma 1, $d(w_2, P) + d(u_s, P) \leq s$. $d(u_s, C_1)+d(w_2, C_1) \geq n+k-1-s-\sum_{i=2}^{k}|C_i|-(k-1) = |C_1|$. So $d(u_s, C_1) \geq |C_1| - 2 = 1$, a contradiction. So $u_1w_2 \notin E(G)$, $d(u_s, C_1) \leq 1$. By Lemma 1, $d(w_2, P) + d(u_s, P) \leq s - 1$. Then $d(u_s, C_1) \geq n + k - 1 - (s - 1) - \sum_{i=2}^{k}|C_i| - (k - 1) - 2 \geq |C_1| + 1 - 2 = 2$, a contradiction.

Case 2. $d(u_i, C_j) \geq 2$ for some $i, j (i \leq s, j \leq k)$.

Claim 2.1. $d(u_i, C_j) \leq 1$ for any $i, j, i \leq s, j \in I_m$, i.e., $|C_j| = 3$.

Suppose to the contrary, there is some i, j satisfying $d(u_i, C_j) \geq 2, i \leq s, j \in I_m$. We assume that $d(u_1, C_1) \geq 2$. Then $d(u_s, C_1) = 0$. $d(u_2, C_1) = 0$. Let $C_1 = w_1v_1w_2w_1$. By Lemma 1, $d(u_2, P) + d(w_2, P) \leq s$, $d(v_1, P) + d(u_s, P) \leq s$. So $d(u_2, T - C_1) + d(v_1, T - C_1) + d(u_s, T - C_1) + d(w_2, T - C_1) = d(u_2) + d(v_1) +$

$d(u_s)+d(w_2)-[d(u_2,P)+d(w_2,P)]-[d(u_s,P)+d(v_1,P)]-d(v_1,C_1)-d(u_s,C_1)-d(u_2,C_1)-d(w_2,C_1) \geq 2n+2k-2-2s-4 = 2(n-s-|C_1|)+2k = 2\sum_{i=2}^{k}(|C_i|+1)+2$ implying some C_i in $T-C_1$, say $i=2$, satisfying $d(u_2,C_2)+d(v_1,C_2)+d(u_s,C_2)+d(w_2,C_2) \geq 2(|C_2|+1)+1 = 2|C_2|+3$. If $|C_2|=3$, then $d(v_1,C_2)+d(w_2,C_2) \leq 5$, $d(u_2,C_2)+d(u_s,C_2) \geq 4$. So there exists a hamilton cycle C_2' of $G[V(C_2)\cup(V(P)-\{u_1\})]$, and a hamilton cycle C_1' of $G[V(C_1)\cup\{u_1\}]$, we get the desired cycles $C_1', C_2', C_3, \ldots, C_k$, a contradiction. So $|C_2|=4$. $d(u_2,C_2)+d(v_1,C_2)+d(u_s,C_2)+d(w_2,C_2) \geq 11$. If $d(v_1,C_2)=3$, $d(w_2,C_2)=4$, then we get two triangles $C_1'=v_1x_2x_3v_1$, $C_2'=w_2x_1v_2w_2$, a contradiction to (ii). So $d(v_1,C_2)+d(w_2,C_2) \leq 6$, $d(u_2,C_2)+d(u_s,C_2) \geq 5$. Then there exists a hamilton cycle C_1' of $G[V(C_1)\cup\{u_1\}]$, and a hamilton cycle C_2' of $G[V(C_2)\cup(V(P)-\{u_1\})]$, a contradiction. As claimed.

Therefore by Claim 2.1, $d(u_i,C_j) \geq 2$ for some $i, j, i \leq s, j \in \{1,\ldots,k\}-I_m$, i.e., $|C_j|=4$.

If $d(u_i,C_j) \neq 3$ for any $i, j, i \leq s, j \in \{1,\ldots,k\}-I_m$.

We may assume that $d(u_1,C_1)=2$. Then $d(u_2,C_1) \leq 2$, $d(u_s,C_1) \leq 2$. So there exists some vertex x in C_1 such that $u_1x \notin E(G)$, $u_2x \notin E(G)$,$u_sx \notin E(G)$, but $u_1x^- \in E(G)$. By Lemma 1, $d(u_2,P)+d(x,P) \leq s-1$, $d(u_s,P)+d(x,P) \leq s-1$. So $d(u_2,T-C_1)+d(u_s,T-C_1)+2d(x,T-C_1) \geq 2n+2k-2-2(s-1)-4-4 = 2(n-s-|C_1|)+2k = 2\sum_{i=2}^{k}(|C_i|+1)+2$ yielding that for some $i \geq 2$, say $i=2$, satisfying $d(u_2,C_2)+d(u_s,C_2)+2d(x,C_2) \geq 2(|C_2|+1)+1 = 2|C_2|+3$. If $|C_2|=3$, then $d(x,C_2) \leq 3$, $2d(x,C_2) \leq 6$, $d(u_2,C_2)+d(u_s,C_2) \geq 3$. So $d(u_2,C_2) \geq 2$ or $d(u_s,C_2) \geq 2$, a contradiction to Claim 2.1. So $|C_2|=4$. By the maximality of m, $d(x,C_2) \leq 3$, $2d(x,C_2) \leq 6$, $d(u_2,C_2)+d(u_s,C_2) \geq 5$. So $d(u_2,C_2) \geq 3$ or $d(u_s,C_2) \geq 3$, a contradiction to assumption.

Hence $d(u_i,C_j)=3$ for some $i, j, i \leq s, j \in \{1,\ldots,k\}-I_m$.

We may assume that $d(u_1,C_1)=3$. Say $C_1=w_1w_2w_3v_1w_1$. Then by the maximality of m, $N(u_1,C_1)=\{w_1,w_2,w_3\}$. $d(u_2,C_1)=0$, $d(u_s,C_1)=0$. By Lemma 1, $d(u_2,P)+d(v_1,P) \leq s-1$, $d(u_s,P)+d(v_1,P) \leq s-1$. Hence $d(u_2,T-C_1)+d(u_s,T-C_1)+2d(v_1,T-C_1) \geq 2n+2k-2-2s+2-4 = 2\sum_{i=2}^{k}(|C_i|+1)+6$ yielding that for some $i \geq 2$, say $i=2$, satisfying $d(u_2,C_2)+d(u_s,C_2)+2d(v_1,C_2) \geq 2(|C_2|+1)+1 = 2|C_2|+3$. If $|C_2|=3$, then $d(v_1,C_2) \leq 2$,$(v_1v_2 \notin E(G))$, $2d(v_1,C_2) \leq 4$, $d(u_2,C_2)+d(u_s,C_2) \geq 5$. So $d(u_2,C_2)=3$ or $d(u_s,C_2)=3$, a contradiction to Claim 2.1. So $|C_2|=4$. By the maximality of m, $d(v_1,C_2) \leq 2$, $2d(v_1,C_2) \leq 4$, $d(u_2,C_2)+d(u_s,C_2) \geq 7$. So $d(u_2,C_2)=4$ or $d(u_s,C_2)=4$, a contradiction to the maximality of m. This completes the proof of Theorem 12.

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